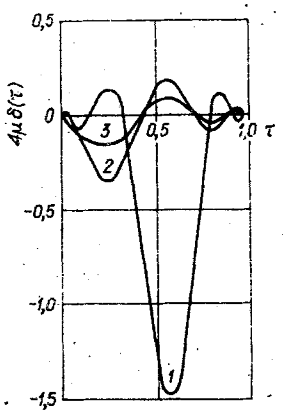

Fig. 1

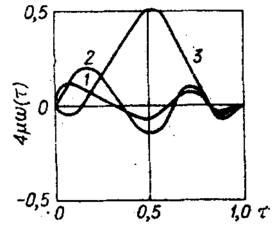

Fig. 2

The results of the calculations are presented in Figs. 1 and 2. The quantities $\delta(\tau)$ and $\omega(\tau)$ multiplied by 4μ are indicated along the vertical axes and, in addition, $\tau = t(1 + t)^{-1}$. This substitution permits studying the behavior of the functions indicated over the entire time interval. Curves 1-3 correspond to V = 1, 10, and 20 m/sec.

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STABILITY OF WELL WALLS

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UDC 622.24.026.3.001 + 539.3

1. Introduction. The scientific—technical problem of superdeep drilling is extremely difficult. Modern technology for constructing wells [1-3] consists of repeating the following cycle many times: drilling the bottom hole of a well with a special bit — extracting pieces of the fractured rock with a flushing liquid — wear or breakage of the drilling equipment and its replacement, usually including raising and lowering operations for the entire column of drillpipes.

Reinforcement of the walls of superdeep wells (exceeding 6 km) with casing columns becomes technically very complicated due to the loss of stability of the well walls, their collapse, and as a result, the large increase in the transverse cross section. In this case, the presence of the hydraulic pressure of the column of washing liquid serves as an important stabilizing factor. Clay and other additives in this liquid, by plugging pores, lead to the formation of a dense crust on the walls, hereby hermetically sealing the well. In what follows, we examine only vertical wells that are not protected by a casing column near the bottom hole at a distance, at least, of the order of 100 diameters of the well. Percolation of the liquid into the rock is neglected.

A very important factor under these conditions is the pressure from the above-lying rock. Considerable technological difficulties in superdeep drilling also arise from the increase in temperature (approximately by 20°C for each kilometer).

2. Local Instability of the Walls of a Circular Well. The well is a cylindrical cavity, $r < r_0$, $0 < z < H$ in the earth's crust $z < H$, where r and z cylindrical coordinates (z coin-

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cides with the axis of the well, Fig. 1). The bottom hole of the well (the end-face of the cylinder is at $z = 0$) breaks down under the action of the teeth on the drill bit, which is pressed into and rotates around the axis of the well. The walls of the well and of the bottom hole break down at great depths under the rock pressure, so that the shape of the well turns out to be very different from the shape indicated. We shall examine this process of natural breakdown far from the bottom hole (in practice, at distances greater than $5r_0$).

We shall denote by q the unperturbed vertical rock pressure, and by ηq the unperturbed lateral rock pressure; the lateral thrust coefficient η can be less than or greater than one depending on the geotectonic conditions.

Far from the well we have

$$\sigma_z = -q,\ \sigma_r = \sigma_\theta = -\eta q \qquad (q > 0). \tag{2.1}$$

The quantity q equals ρgH, where g is the acceleration of gravity, ρ is the average density of above-lying rocks, H is the distance of the point being examined from the earth's surface (on the average $\rho g \sim 3.5$ g/cm³).

We shall examine the initial circular contour of the well, created by the drilling instrument (its transverse cross section is shown in Fig. 1b). Some point O on the wall of this well will be subjected to triaxial compression by the stresses

$$\sigma_z = -q,\ \sigma_r = -p,\ \sigma_\theta = p - 2\eta q. \tag{2.2}$$

Here p is the hydrostatic pressure of the liquid in the well ($p \sim \rho_H gH$, where for water $\rho_H \sim 1$ g/cm³, while for clay solutions $\rho_H g$ can attain 2.5 g/cm³). The peripheral stress σ_θ is obtained from a solution of the corresponding problem in the theory of elasticity for a circular opening [4].

Two cases are possible: $|\sigma_\theta| > |\sigma_z| > |\sigma_r|$, when $p - 2\eta q < -q$, i.e., $(2\eta - 1)q > p$; $|\sigma_z| > |\sigma_\theta| > |\sigma_r|$, when $p - 2\eta q > -q$, i.e., $(2\eta - 1)q < p$.

In these cases, the nature of the local breakdown at the point O will be different and the process of cavity formation will occur differently.

The criterion for local fracture can be represented as a surface $f(\sigma_z, \sigma_r, \sigma_\theta) = 0$, encompassing the origin of coordinates in the space $\sigma_z\sigma_r\sigma_\theta$. In the region of compressive stresses of interest here $\sigma_z < 0$, $\sigma_r < 0$, $\sigma_\theta < 0$ for $|\sigma_z| > |\sigma_r|$ and $|\sigma_\theta| > |\sigma_r|$ the surface can be represented [5, 6] as follows:

$$\text{for } |\sigma_\theta| > |\sigma_z| > |\sigma_r| \quad \sigma_\theta = -\sigma_c + \delta(\sigma_z + \sigma_r), \tag{2.3}$$

$$\text{for } |\sigma_z| > |\sigma_\theta| > |\sigma_r| \quad \sigma_z = -\sigma_c + \delta(\sigma_\theta + \sigma_r).$$

Here δ and σ_c are empirical constants, chosen so as to describe best the experimental data in the range of stresses studied.

The number δ, similar to the Poisson coefficient, satisfies the inequality $0 \leqslant \delta \leqslant 1/2$. Without making a large error, it can be taken as equal to 1/2 (this corresponds to the experimental fact that the strength under hydrostatic compression, when $\sigma_r = \sigma_z = \sigma_\theta$, exceeds by many times the strength under uniaxial compression).

Substituting (2.2) into (2.3), we find the following condition for local fracture at the point O:

$$\text{for } 2\eta q - p > q > p \quad (2\eta - \delta)q = \sigma_c + p(1 + \delta), \tag{2.4}$$

$$\text{for } (2\eta - 1)q < p < \eta q \quad q(1 - 2\delta\eta) = \sigma_c.$$

For $(2\eta - 1)q > p$, the displacement at the point O of the well wall at the time of local fracture will occur along the surface parallel to the z axis, and for $(2\eta - 1)q < p$ along the surface inclined to the z axis at some angle and parallel to the tangent to the circular contour of the well at the point O.

We shall investigate the stability of the shape of the circular well contour with respect to small perturbations of the contour shape, which are completely unavoidable in the process of constructing the well. For example, assume that near the point O there is a very small depression (Fig. 1b). This depression leads to additional concentration of stresses

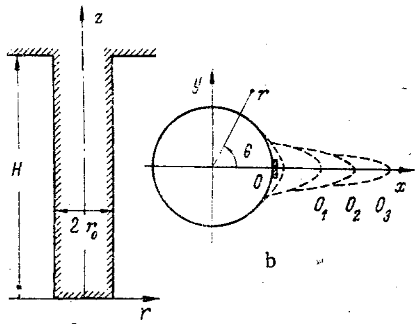

Fig. 1

in the vicinity of this point, as a result of which, as the rock pressure increases the local fracture in it occurs earlier than at other points of the circular well contour. Evidently, in view of the continuous increase in the concentration of stress in the growing depression the process of local fracture will be a self-sustaining process until the system passes into a stable equilibrium state (i.e., when the well assumes a new shape, stable relative to the unavoidable small perturbations).

We emphasize that the rocks studied have the following property: formation of a region with a limiting state indicates separation into small disconnected particles at all points of this region; these particles are flushed out by the flushing drilling liquid, until the region of the limiting state degenerates into some surface, which will be the boundary of the body. Further flushing of rock is impossible, since the remaining volume will be continuous and elastic. For this reason, under unchanged external conditions, the limiting boundary of the body obtained will be unchanged and it is natural to refer to it as the equilibrium shape of the body.

According to (2.4), the hydrostatic pressure of the fluid affects only the stability of the circular walls and the cavern formation of the first kind; cavern formation of the second kind does not depend on the pressure of the liquid in the circular well.

Taking into account the stratified (layered) structure of the earth's crust, it is not difficult to imagine that the phenomenon of loss of stability and cavern formation described above can occur also at small depths in layers with little strength. For this reason, with optimal control of drilling, it should be kept in mind that by choosing the control parameter p, it is possible to avoid cavern formation of the first kind, for which $p(1 + \delta) > (2\eta - \delta)q - \sigma_c$ must be satisfied in the corresponding layer. Cavern formation of the second kind, according to the second relation (2.4), is practically an uncontrollable process (only the formation of a clay crust on the walls of the well under the action of the drilling liquid has any effect on σ_c and δ of the rock).

3. Formulation of the Problem of Equilibrium Shapes of Elastic Bodies. We shall examine the phenomenon of cavern formation of the first kind, occurring under conditions close to planar deformation in the xy plane of the transverse cross section of the well. We shall study the possible equilibrium states. The equilibrium shape of an elastic body is a shape for which all boundary points of the body have the same possibility for failure, i.e., when the stress concentration is the same on all points of the boundary. These are so-called "equal-strength" bodies [7, 8].

We shall denote by L the unknown "equal-strength" contour of an elastic body in the complex $z = x + iy$ plane (Fig. 2). We have the following boundary conditions:

on contour L

$$\sigma_n = -p, \ \tau_{nt} = 0, \qquad \sigma_t = -\sigma = \text{const}; \tag{3.1}$$

for $z \to \infty$

$$\sigma_x = \sigma_y = -\eta q, \ \tau_{xy} = 0. \tag{3.2}$$

Here t and n are the tangential and normal to the contour (forming a right-handed system nt). Taking into account the noncircular shape of the cavity, according to (2.3), we have

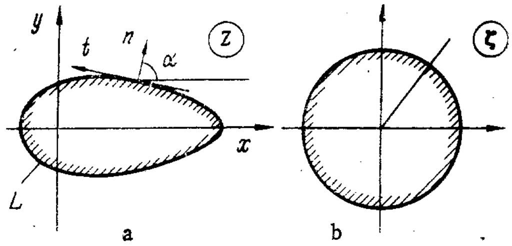

Fig. 2

$$
\begin{gathered}
\text{for } |\sigma_t| \geqslant |\sigma_z| \geqslant |\sigma_n| \text{ on cavern walls, i.e.,} \\
\text{for } \sigma \geqslant |q + \nu(p + \sigma - 2\eta q)| \geqslant p, \\
\sigma_t = -\sigma_c + \delta(\sigma_z + \sigma_n), \\
\sigma = [1/(1 - \nu\delta)][\sigma_c + \delta p(1 + \nu) + \delta q(1 - 2\nu\eta)]; \\
\text{for } |\sigma_z| \geqslant |\sigma_t| \geqslant |\sigma_n| \text{ on cavern walls, i.e.,} \\
\text{for } |q + \nu(p + \sigma - 2\eta q)| \geqslant \sigma \geqslant p, \\
\sigma_z = -\sigma_c + \delta(\sigma_n + \sigma_t), \\
\sigma = -p + (q - \sigma_c - 2\nu\eta q)/(\delta - \nu)
\end{gathered} \tag{3.3}
$$

(ν is Poisson's coefficient). It was assumed here that the stress field sought in the body is a sum of an unperturbed uniform field $\sigma_n^0 = \sigma_t^0 = -\eta q$, $\sigma_z^0 = -q$ and the nonuniform field σ_{ik} is caused by the shaft, under conditions of planar deformation, so that $\sigma_z' = \nu(\sigma_n' + \sigma_t')$.

We shall represent the stress components in terms of the Kolosov–Muskhelishvili potentials $\Phi(z)$ and $\Psi(z)$:

$$
\begin{gathered}
\sigma_x + \sigma_y = 4 \operatorname{Re} \Phi(z) \quad (z = x + iy), \\
\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{z}\Phi'(z) + \Psi(z)].
\end{gathered}
$$

According to (3.2), we have

$$
\text{for } z \to \infty \quad \Phi(z) = -(1/2)\eta q + O(z^{-2}), \quad \Psi(z) = O(z^{-2}). \tag{3.4}
$$

Using the well-known relations

$$
\begin{gathered}
\sigma_t + \sigma_n = \sigma_x + \sigma_y, \\
\sigma_t - \sigma_n + 2i\tau_{tn} = e^{2i\alpha}(\sigma_y - \sigma_x + 2i\tau_{xy}),
\end{gathered}
$$

where α is the angle between n and x (measured from x to n), the boundary conditions (3.1) on the countour L can be written in the form

$$
\begin{gathered}
4 \operatorname{Re} \Phi(z) = -\sigma - p, \\
\bar{z}\Phi'(z) + \Psi(z) = (1/2)(p - \sigma)e^{-2i\alpha}.
\end{gathered} \tag{3.5}
$$

4. Solution of the Boundary-Value Problem within the Class of Bounded Potentials. We first seek the solution of this problem in the class of everywhere bounded potentials $\Phi(z)$ and $\Psi(z)$ and in the class of contours L with infinite branches. In this class of functions, according to the first boundary condition (3.5), the potential $\Phi(z)$ will be everywhere constant:

$$
\Phi(z) = -(1/4)(\sigma + p), \tag{4.1}
$$

and, in addition, from the conditions at infinity (3.4)

$$
\sigma = 2\eta q - p
$$

and, according to Eq. (3.3), the stress state at the walls of the "equal-strength" well sought will be as follows:

$$
\sigma_n = -p, \quad \sigma_t = -2\eta q + p, \quad \sigma_z = -q,
$$

where

$$(2\eta-\delta)q=\sigma_c+p(1+\delta) \quad \text{for } 2\eta q-p\geqslant q\geqslant p,$$
$$(1-2\eta\delta)q=\sigma_c \quad \text{for } q\geqslant 2\eta q-p\geqslant p. \tag{4.2}$$

Substituting (4.1) into (3.5), we obtain

$$2\Psi(z)=(p-\sigma)e^{-2i\alpha} \quad (z\in L). \tag{4.3}$$

Let us make a conformal transformation of the exterior of the contour L in the z plane on the exterior of the unit circle $|\zeta|>1$ in the parametric plane ζ with the help of the analytic function $\omega(\zeta)$, which must be determined: $z=\omega(\zeta)$.

Let us determine $e^{2i\alpha}$. We shall give the increment at the point z in the direction of the normal to the contour L (see Fig. 2)

$$dz=e^{i\alpha}|dz|. \tag{4.4}$$

The corresponding point in the plane ζ, in view of the conformal nature of the transformation, will be displaced along the radius

$$d\zeta=\zeta|d\zeta|. \tag{4.5}$$

With the help of (4.4) and (3.5), we find

$$e^{2i\alpha}=\left(\frac{dz}{|dz|}\right)^2=\left(\frac{\omega' d\zeta}{|\omega'||d\zeta|}\right)^2=\frac{\zeta^2\omega'(\zeta)}{\overline{\omega'(\zeta)}}. \tag{4.6}$$

The boundary-value problem (4.3) on the ζ plane is written in the form

$$2\psi(\zeta)=(p-\sigma)\overline{\omega'(\zeta)} \quad \text{for } |\zeta|=1. \tag{4.7}$$

Here $\psi(\zeta)=\zeta^2\omega'(\zeta)\Psi[\omega(\zeta)]$, for $\zeta\to\infty$ $\psi(\zeta)=O(1)$, $\omega'(\zeta)=O(1)$.

Let the contour L sought have 2n infinite branches, which correspond to 2n simple poles of the functions $\omega'(\zeta)$ and $\psi(\zeta)$ at the points $\zeta=\zeta_k$ and $\zeta=-\zeta_k$:

$$\zeta_k=e^{i\varphi_k}\,(k=1,2,3,\ldots,n).$$

The general solution of the boundary-value problem (4.7) in the class of functions indicated has the form

$$\omega'(\zeta)=\sum_{k=1}^{n}\left(A_k\frac{\zeta+\zeta_k}{\zeta-\zeta_k}+B_k\frac{\zeta-\zeta_k}{\zeta+\zeta_k}\right)+C_0,$$

$$\psi(\zeta)=-\frac{1}{2}(p-\sigma)\left[-\overline{C}_0+\sum_{k=1}^{n}\left(\overline{A}_k\frac{\zeta+\zeta_k}{\zeta-\zeta_k}+\overline{B}_k\frac{\zeta-\zeta_k}{\zeta+\zeta_k}\right)\right],$$

where C_0, A_k, and B_k are arbitrary complex constants.

Since the function $\omega(\zeta)$ must be single-valued in circumscribing the unit circle, $B_k=A_k$.

Integrating, we find the function $\omega(\zeta)$

$$\omega(\zeta)=C\zeta+2\sum_{k=1}^{n}\zeta_kA_k\ln\frac{\zeta-\zeta_k}{\zeta+\zeta_k}\ \left(C=C_0+2\sum A_k\right). \tag{4.8}$$

The function $\ln[(\zeta-\zeta_k)/(\zeta+\zeta_k)]$ is single-valued in the plane ζ with a cut, connecting the points $\zeta=\zeta_k$ and $\zeta=-\zeta_k$ inside the unit circle.

Let the contour L be symmetrical relative to the real axis. We shall assume that the real axis of the ζ plane goes over to the real axis of the z plane in the vicinity of a point at infinity. From here it follows that the value of C is real; real and imaginary ζ_k correspond to real coefficients A_k, while each pair of conjugate poles ζ_k and $\overline{\zeta}_k$ correspond to a pair of conjugate complex coefficients A_k and $\overline{A}_k$.

Thus, the family of contours L sought, defined by the functions $\omega(\zeta)$ in (4.8), depends on 2N arbitrary constants A_k and ζ_k, where the same index k corresponds to four complex poles (ζ_k, $\overline{\zeta}_k$, $-\zeta_k$, $-\overline{\zeta}_k$) and a pair of real (ζ_k, $-\zeta_k$) or imaginary (ζ_k, $\overline{\zeta}_k$) poles. The transformation, effectuated by this function, is not a single sheet transformation: the exterior of

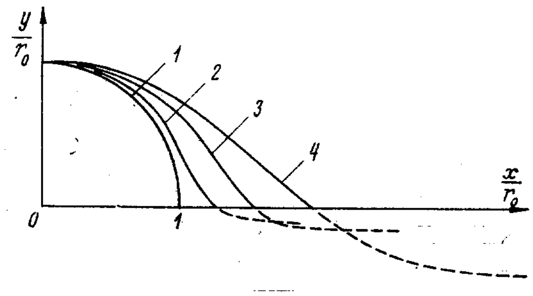

Fig. 3

the circle $|\zeta| > 1$ corresponds to a two-sheet Riemann surface in the z plane with symmetrical infinite branches on the second sheet.

Let us examine the physical meaning of the solution obtained as an example. Assume that there are only two poles: at the points $\zeta = \zeta_1 = 1$ and $\zeta = -\zeta_1 = -1$. In this case, according to (4.8), the transformation is defined by the function

$$\omega(\zeta) = C\left(\zeta - \lambda \ln \frac{\zeta - 1}{\zeta + 1}\right),$$

where C and λ are arbitrary positive parameters. In this case, the unit circle $\zeta = e^{i\varphi}$ goes over into the contour L with self-intersecting branches, whose equation has the form

$$\begin{cases} x = C\left[\cos\varphi - \lambda \ln\left(\alpha \tan\frac{\varphi}{2}\right)\right], \\ y = C\left(\sin\varphi - \frac{\pi}{2}\alpha\lambda\right) (\lambda > 0, C > 0), \end{cases} \qquad (4.9)$$

where $\alpha = 1$ for $0 < \varphi < \pi$ and $\alpha = -1$ for $\pi < \varphi < 2\pi$.

It is natural to assume that for $\varphi = \pm\pi/2$ $y = \pm r_0$, from which we have

$$C = \frac{r_0}{1 - \frac{1}{2}\pi\lambda}.$$

Therefore, the parameter λ must satisfy the inequality $0 < \lambda < 2/\pi$.

The x and y axes are the symmetry axes for the contour L. Figure 3 presents the family of equilibrium contours L, constructed in the right half plane according to Eqs. (4.9) with values $\lambda = 0$, 0.05, 0.1, and 0.2 (curves 1-4, respectively). The self-intersecting branches are shown by the dashed lines; they are located on the second sheet of the two-sheet Riemann surface z.

The solution constructed describes the successive growth of the cavern from a circular opening as a result of infinitely small starting failures at the points $y = 0$, $x = \pm r_0$ of the opening. The presence of regions of self-intersection shows that there are always zones of the supercritical state, not flushed out by the liquid, on the continuation of the cavern into the body. The equilibrium contours of the cavern obtained will, evidently, be locally unstable and can transform randomly into any of the equilibrium shapes, described by the general solution (4.8). According to this solution, there exist infinitely many forms of equilibrium caverns with the oddest configurations and with an arbitrary number of "canyons." All these shapes are locally unstable.

We note that the two-sheet Riemann surface was already used previously as a physical space in some hydrodynamic papers. This is primarily the Efros–Dzhil'barg–Rocca model in the theory of cavitation [9, 10]. The parameter λ in solution (4.9) plays the role of "time," so that it may be assumed that this solution describes the growth of a finite perturbation (see, e.g., [11] on the growth of a "tongue" of liquid from a small perturbation of an unstable equilibrium form of two liquids).

5. Solution of the Boundary-Value Problem on the Class of Unfounded Potentials. We shall now study the solution of the starting problem (3.5) in the class of bounded contours

L with return points and in the class of unbounded at these points potentials $\Phi(z)$ and $\Psi(z)$. At the continuation of return points in the body, the stresses are infinite and for this reason in the vicinity of such a point there is a zone of supercritical state, where the rock is fractured. It follows from the preceding discussion that the formation of such zones is unavoidable in the growth of caverns.

In what follows, it is assumed that the characteristic linear size of regions of the limiting and superlimiting state in the vicinity of the return point is small compared to the characteristic linear dimension of the cavern. This assumption is similar to the assumption of a fine structure in the theory of brittle cracks, and it permits using elastic solutions with integrable singularities.

Assume that the contour sought L has n return points, which in the parametric plane ζ corresponds to the points $\zeta = \zeta_k = e^{i\varphi_k}$ on the unit circle, where k = 1, 2, ..., n. The root singularity of the complex potentials at the return point in the z plane corresponds to a simple pole at the corresponding point ζ_k in the ζ plane, since

$$\omega'(\zeta_k) = 0, \text{ for } \zeta \to \zeta_k \ \omega(\zeta) = \omega(\zeta_k) + O[(\zeta - \zeta_k)^2].$$

Solving the Dirichlet problem for the exterior of the unit circle $|\zeta| > 1$ in the class of functions indicated, according to the first boundary condition (3.5), we obtain

$$\varphi(\zeta) = -\frac{1}{4}(\sigma + p) + \sum_{k=1}^{n} A_k \frac{\zeta + \zeta_k}{\zeta - \zeta_k}, \ \varphi(\zeta) = \Phi[\omega(\zeta)], \tag{5.1}$$

where $A_1, A_2, \ldots, A_n$ are some arbitrary real constants, satisfying, according to (3.4), the following relations:

$$\sum_{k=1}^{n} A_k = \frac{1}{4}(\sigma + p - 2\eta q), \ \sum_{k=1}^{n} A_k\zeta_k = 0.$$

Let us introduce the new functions

$$\chi(\zeta) = \frac{1}{2}(p - \sigma)\,\omega'(\zeta) + 2\omega(\zeta)\sum_{k=1}^{n} \frac{A_k\zeta_k}{(\zeta - \zeta_k)^2}, \tag{5.2}$$

analytic in the region exterior to the unit circle $|\zeta| > 1$ and bounded at $\zeta \to \infty$.

Substituting $\varphi(\zeta)$ from (5.1) into the second condition (3.5) and using (4.6), we obtain the following boundary-value problem:

$$\psi(\zeta) = \overline{\chi(\zeta)} \text{ for } |\zeta| = 1,$$

where

$$\psi(\zeta) = \zeta^2\omega'(\zeta)\Psi[\omega(\zeta)].$$

The general solution of this boundary-value problem in the class of functions bounded at infinity and, according to (5.2), having second-order poles at the points $\zeta = \zeta_k$, has the form

$$\begin{aligned} \psi(\zeta) &= -\sum_{k=1}^{n} \bar{B}_k \frac{\zeta + \zeta_k}{\zeta - \zeta_k} + \sum_{k=1}^{n} \bar{C}_k \left(\frac{\zeta + \zeta_k}{\zeta - \zeta_k}\right)^2, \\ \chi(\zeta) &= \sum_{k=1}^{n} B_k \frac{\zeta + \zeta_k}{\zeta - \zeta_k} + \sum_{k=1}^{n} C_k \left(\frac{\zeta + \zeta_k}{\zeta - \zeta_k}\right)^2, \end{aligned} \tag{5.3}$$

where B_k and C_k are some arbitrary complex constants.

Since the function $\chi(\zeta)$ has been found, relation (5.2) represents a linear differential first-order equation for the function sought $\omega(\zeta)$. An investigation of the analytic nature of this solution shows that for $\sigma \neq p$, at the points $\zeta = \zeta_k$ the function $\omega(\zeta)$ has an essential singularity of the type $(1/(\zeta - \zeta_k)$. In this problem, this singularity has no physical significance, so that we must set

$$\sigma = p. \tag{5.4}$$

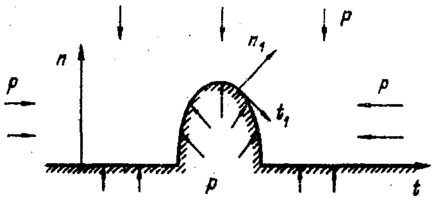

Fig. 4

In addition, the stress state on the cavity wall, according to (3.3), will be as follows:

$$\sigma_n = \sigma_t = -p,\ \sigma_z = -q + 2\nu(\eta q - p),$$

and, in addition, for $q(1 - 2\nu\eta) > p(1 - 2\nu)$

$$q(1 - 2\nu\eta) + 2p(\nu - \delta) = \sigma, \tag{5.5}$$

Based on (5.2)-(5.4), the function sought $\omega(\zeta)$ has the form

$$\omega(\zeta) = \left[\sum_{k=1}^{n} \frac{2A_k\zeta_k}{(\zeta - \zeta_k)^2}\right]^{-1} \sum_{k=1}^{n} B_k' \frac{\zeta + \zeta_k}{\zeta - \zeta_k}\left(1 + C_k' \frac{\zeta_k}{\zeta - \zeta_k}\right), \tag{5.6}$$

where B_k' and C_k' are arbitrary complex constants. The constants A_k, B_k', C_k', and ζ_k must satisfy only the conditions $\omega'(\zeta_k) = 0$ and the condition for the conformal transformation $|\omega'(\zeta)| > 0$ for $|\zeta| > 1$, but other than that they are arbitrary. For this reason, there are infinitely many forms of equilibrium caverns with return points.

We shall show that all of these states will be locally unstable relative to any infinitely small smooth changes in the shape of the contour. In the vicinity of an infinitely small perturbation in the shape of the contour ("tongue" in Fig. 4), based on the "microscope principle" [6], it may be assumed that in the unperturbed state the body occupied the upper half-space $n > 0$. It is necessary to determine the stress field in the upper half-space with a cutout having a smooth shape (without corners and return points), if the stresses $\sigma_n = \sigma_t = -p$ act at infinity, while the loads $\sigma_{n_1} = -p$, $\tau_{t_1 n_1} = 0$, where n_1 and t_1 are the normal and tangential to the perturbed boundary contour, are applied to the entire changed boundary. The stress fields sought at all points of the body, evidently, will represent hydrostatic compression with magnitude p; in particular, on the cutout, as in the unperturbed state, $\sigma_{n_1} = \sigma_{t_1} = -p$ as before. Therefore, infinitely small smooth perturbations of equilibrium forms constructed at this point do not lead to stress concentration and progressive local fracture, i.e., the equilibrium forms obtained will be locally stable.

Without analyzing the more general solution (5.6), we shall present only some symmetrical particular cases.

Two Return Points. In this case,

$$\varphi(\zeta) = -\frac{1}{2}\eta q + \frac{p - \eta q}{\zeta^2 - 1},\ \omega(\zeta) = \frac{1}{2} l\left(\zeta + \frac{1}{\zeta}\right)$$

and the contour L represents a crack like cavity of length $2l$.

Four Symmetrical Return Points. In this case,

$$\varphi(\zeta) = -\frac{1}{2}\eta q + \frac{p - \eta q}{\zeta^4 - 1},\ \omega(\zeta) = \frac{3}{2} r_0\left(\zeta + \frac{1}{3\zeta^3}\right)$$

and the contour L is an astroid

$$x^{2/3} + y^{2/3} = (2r_0)^{2/3},$$

whose shortest distance from the origin of coordinates equals r_0.

Symmetrical Countour with 2n Return Points. In this case,

$$\varphi(\zeta) = -\frac{1}{2}\eta q + \frac{p - \eta q}{\zeta^{2n} - 1},\ \omega(\zeta) = \frac{2n - 1}{2n - 2} r_0\left[\zeta + \frac{1}{(2n - 1)\zeta^{2n-1}}\right]$$

and the contour L is a hypocycloid

$$x = \frac{2n-1}{2n-2} r_0 \left[\cos\varphi + \frac{1}{2n-1}\cos(2n-1)\varphi\right],$$
$$y = \frac{2n-1}{2n-2} r_0 \left[\sin\varphi - \frac{1}{2n-1}\sin(2n-1)\varphi\right] (0 < \varphi < 2\pi),$$

whose shortest distance from the origin of coordinates equals r_0. It is natural to assume that the quantity r_0 corresponds to the radius of the initial circular well, so that the contour of the stable cavern formed for $n > 1$ touches the starting circular contour at 2n points.

The potential $\psi(\zeta)$ is determined from the functions $\varphi(\zeta)$ and $\omega(\zeta)$ found as follows:

$$\psi(\zeta) = -\zeta^2\varphi'(\zeta)\overline{\omega}(1/\zeta).$$

6. Limiting Well Depth. The results obtained above lead to the following picture of the development of caverns from the starting wells. In this formulation of the problem, the well depth H (the loading parameter) will be the analog of time. For fixed parameters σ_c, η, δ, ν, ρg, $\rho_H g$, the cavern growth process is determined by the quantity H. We shall examine this process.

According to Eqs. (4.2), (5.5), the following three variants of fracture are possible:

variant I

$$(2\eta - \delta)\rho g H - (1 + \delta)(\rho_H g H + p_a) = \sigma_c; \tag{6.1}$$

variant II

$$(1 - 2\eta\delta)\rho g H_* = \sigma_c; \tag{6.2}$$

variant III

$$(1 - 2\nu\eta)\rho g H + 2(\nu - \delta)(\rho_H g H + p_a) = \sigma_c \tag{6.3}$$
$$(q = \rho g H,\ p = \rho_H g H + p_a).$$

Here p_α is the additional pressure of the drilling liquid on the earth's surface.

The variants correspond to the following values of H:

$$H_{\rm I} = \frac{\sigma_c + p_a(1+\delta)}{\rho g(2\eta - \delta) - \rho_H g(1+\delta)}; \tag{6.4}$$

$$H_{\rm II} = \frac{\sigma_c}{\rho g(1 - 2\eta\delta)}; \tag{6.5}$$

$$H_{\rm III} = \frac{\sigma_c - 2p_a(\nu - \delta)}{\rho g(1 - 2\nu\eta) + 2\rho_H g(\nu - \delta)}. \tag{6.6}$$

Only positive values of H have any physical meaning. For this reason, for negative values of $H_{\rm I}$, $H_{\rm II}$ or $H_{\rm III}$, the corresponding fracture variant is not realized.

The formation of the cavern, evidently, begins at depth $H = H_*$, equal to

$$H_* = \min(H_{\rm I}, H_{\rm II}). \tag{6.7}$$

In the process of its development at $H = H_*$, the cavern goes from an initial circular form through a set of continuously changing and locally unstable equilibrium forms, described by the solutions in Sec. 4. The rate of growth of the cavern at this stage is determined by the velocity of the drilling liquid flushing out the fractured particles. In this formulation of the problem, this velocity can be assumed to be infinite. For large deviations from the initial circular form, the unstable nonequilibrium forms with infinite branches (Sec. 4) are no longer real due to the presence of large self-compression zones. It is natural to assume that the final stage of the development of the cavern at $H = H_*$ will be locally stable equilibrium forms with return points (Sec. 5). Evidently, this stage will be stable and in equilibrium as a whole, if $H_{\rm III} > H_* = \min(H_{\rm I}, H_{\rm II})$; in this case, the elastic system will go over into one of the stable states, described by the solutions in Sec. 5. In this case, the cavern formed will not develop until the increasing depth of the well attains the magnitude $H = H_{\rm III}$. Further increase in depth, greater than $H_{\rm III}$, is impossible, since it will be accompanied by continuous and unbounded breakdown of the cavern walls according to criterion (6.3). Thus, in the case examined, the limiting depth of the well $H = H_{**}$ equals H_{III}; a greater drilling depth is impossible for the technology examined.

If, on the other hand, $H_{III} < H_* = \min(H_I, H_{II})$, then the development of the cavern at $H = H_*$ is continuous, since the elastic states in Sec. 5, according to (6.3), in this case will be nonequilibrium, supercritical. In this case, the limiting depth of the well H_{**} equals H_*.

Thus, we have the following general result:

$$H_{**} = \max(H_{III}, H_*) = \max[H_{III}, \min(H_I, H_{II})]. \tag{6.8}$$

Let us give an example. Let $\eta = 1/2$, $\nu = 1/3$, $\delta = 1/2$, $p_\alpha = 0$. In this case, according to (6.4)-(6.6), we have

$$H_I = 2\sigma_c/(\rho - 3\rho_H)g,\ H_{II} = 2\sigma_c/\rho g,\ H_{III} = 3\sigma_c/(2\rho - \rho_H)g.$$

As can be seen, failure variant I can be realized only for $\rho > 3\rho_H$, while variant III can be realized only for $2\rho > \rho_H$. From here, according to Eqs. (6.7) and (6.8), we find the starting depth of the cavern formation and the limiting drilling depth:

$$H_* = \frac{2\sigma_c}{\rho g},$$

$$H_{**} = \begin{cases} \dfrac{2\sigma_c}{\rho g} & \text{for } \rho > 2\rho_H \text{ and } \rho < \dfrac{1}{2}\rho_H, \\ \dfrac{3\sigma_c}{(2\rho - \rho_H)g} & \text{for } 2\rho_H > \rho > \dfrac{1}{2}\rho_H. \end{cases}$$

For example, for $\rho g = 3$ g/cm³, $\rho_H g = 1.5$ g/cm³,

$$H_* = H_{**} = \frac{2}{3}\sigma_c \cdot 10^3 \text{ m}.$$

From here, with $\sigma_c = 30$ kg/mm² (granite), $H_{**} = 20$ km, while for $\sigma_c = 0.003$ kg/mm² (sand) $H_{**} = 2$ m. As is evident, the strength of the rock plays a fundamental role in the design of superdeep wells.

7. Stressed State in the Vicinity of the Bottom Hole of a Well. Let us assume that the well walls are reinforced in the process of drilling right down to the bottom hole (cementing, casing columns, etc.). In this case, in order to forecast the limiting drilling depth, it is necessary to know the stress state in the vicinity of the bottom hole of a reinforced well. This is also necessary in order to study the processes of local fracture under the teeth in the drill bit during drilling, as well as to determine the coefficient of lateral thrust η. It is natural to measure the latter from the measurements of elastic deformations of a core sample when it is extracted from the bottom hole at the earth's surface, and for this it is necessary to fix strain gauges to it before cutting the core sample from the bottom hole.

Thus, let us assume that we have an axisymmetrical cavity in an infinite elastic space, and the shape of the cavity is a semiinfinite circular cylinder with a rounded end-face L (bottom hole of the well). A constant hydrostatic pressure ($\sigma_n = -p$, $\tau_{nt} = 0$) acts on L and the rest of the boundary of the cavity is rigidly fixed, so that all displacements vanish on it. At large distances from the bottom hole constant stresses $\sigma_z = -q$, $\sigma_r = \sigma_\theta = -\eta q$ act (Fig. 5). It is necessary to determine the stresses σ_t and σ_θ on the end-face L.

Let us represent the elastic stress field sought σ_{ik} and displacements u_i in the form

$$\sigma_{ik} = -p\delta_{ik} + \sigma'_{ik}, \quad u_i = u_i^0 + u'_i,$$

where u_i° correspond to the state of hydrostatic compression by pressure p. In this case, the following boundary conditions will be valid for the fields marked with the prime:

$$\sigma'_n = \tau'_{nt} = 0 \text{ on } L,$$
$$\frac{\partial u'_r}{\partial z} = 0, \quad \frac{\partial u'_z}{\partial z} = \frac{1 - 2\nu}{E} p \tag{7.1}$$

where $r = r_0$ on the remaining boundary of the cavity, where E is Young's modulus.

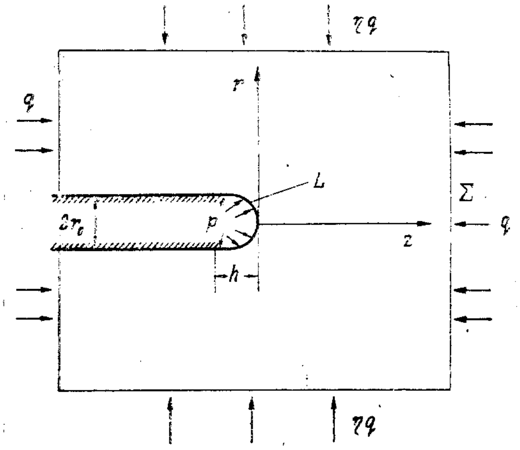

Fig. 5

Far from the cavity, a uniform stress field acts:

$$\sigma_r' = \sigma_\theta' = -\eta q + p, \quad \sigma_z' = -q + p. \tag{7.2}$$

We shall examine a closed surface Σ, consisting of the boundary of the cavity, the cylinder r = R, and the surfaces z = ±H₀, where $R \gg r_0$ and $H_0 \gg r_0$ (Fig. 5).

From the condition of equilibrium of an elastic body within Σ, we have

$$2\int_{-\infty}^{-h} \sigma_{rz}'\big|_{r=r_0} dz = r_0(q-p). \tag{7.3}$$

According to the theory of invariant Γ integrals [6]

$$\int_{\Sigma} (U' n_z - \sigma_n' u_{n,z}' - \sigma_{nt}' u_{t,z}')\, d\Sigma = 0.$$

Here U' is the elastic potential of unit volume for the field with the primes; n and t, directions of the normal and the tangent to the contour of the radial section Σ; n_z, component of the vector normal along the z axis.

From here, based on (7.1)-(7.3), we obtain the equation

$$\frac{1}{2\pi}\int_L U' n_z dL = \left[\int_0^R (U' - \sigma_z' u_{z,z}') r dr\right]_{z=H_0} - \left[\int_{r_0}^R (U' - \sigma_z' u_{z,z}') r dr\right]_{z=-H_0} + \frac{1-2\nu}{2E} p r_0^2 (q-p). \tag{7.4}$$

Using the elementary solutions, we have at z = H₀

$$U' - \sigma_z' u_{z,z}' = \frac{1-\nu}{E}(p-\eta q)^2 - \frac{1}{2E}(p-q)^2,$$

and for z = −H₀

$$U' - \sigma_z' u_{z,z}' = \frac{(\nu - \eta + \nu\eta)^2}{E(1+\nu)} q^2 \left(\frac{r_0}{r}\right)^4 + \frac{1-\nu}{E}(p-\eta q)^4 - \frac{1}{2E}(p-q)^2.$$

Substituting these values into Eq. (7.4), we obtain

$$\int_L U' n_z dL = \frac{\pi}{E} r_0^2 \left\{ -\frac{1}{2}(1-2\nu)p^2 + 2pq(1-\nu)(1-\eta) + q^2\left[\eta^2(1-\nu) - \frac{1}{2} - \frac{1}{1+\nu}(\nu - \eta + \nu\eta)^2\right]\right\} \tag{7.5}$$

Equation (7.5) can be used to estimate the stresses on the bottom hole of the well. For this, we shall first examine the equal-strength bottom hole, for which the following conditions on L will be valid:

$$\sigma_\theta = \sigma_t = -\sigma = \text{const}$$

and, therefore,

$$\sigma_n' = 0, \quad \sigma_\theta' = \sigma_t' = \sigma - p, \quad U' = \frac{1-\nu}{E}(\sigma - p)^2 = \text{const}.$$

From here, with the help of Eqs. (7.5), we obtain the stress σ sought on the equal-strength bottom hole

$$\sigma = p + \frac{1}{\sqrt{1-\nu}}\left\{-\frac{1}{2}(1-2\nu)p^2 + 2pq(1-\nu)(1-\eta) + q^2\left[\eta^2(1-\nu) - \frac{1}{2} - \frac{1}{1+\nu}(\nu - \eta + \nu\eta)^2\right]\right\}^{1/2}.$$

In those cases when the radicand is negative, the equal strength bottom hole does not exist.

Let us examine some limiting particular cases of the existance of an equal-strength bottom hole: for $\eta \gg 1$, $p = 0$ $\sigma = q\eta\sqrt{2\nu/(1+\nu)}$, for $q = 0$ it does not exist, and for $\eta = 0$, $p = 0$ it does not exist.

In the most realistic case $\nu = 1/3$ and $\eta = 1/2$, we have

$$\sigma = p + p\sqrt{\frac{q}{p} - \frac{1}{4} - \frac{1}{2}\left(\frac{q}{p}\right)^2} = p + q\sqrt{\frac{p}{q} - \frac{1}{2} - \frac{1}{4}\left(\frac{p}{q}\right)^2} \tag{7.6}$$

and the equal-strength bottom hole exists in the range

$$\rho_H\left(1 + \frac{\sqrt{2}}{2}\right) > \rho > \rho_H\left(1 - \frac{\sqrt{2}}{2}\right),$$

usually realized in practice.

From physical considerations, for a different form of the bottom hole, the stresses satisfy the inequalities

$$\max(\sigma_\theta, \sigma_t) > \sigma > \min(\sigma_\theta, \sigma_t),$$

so that the quantity σ_{dc} plays the role of an average stress on the bottom hole, especially, since the exact rounded form of the bottom hole, formed during the drilling process, is unknown.

The criterion for failure of the bottom hole under the action of loads p and q can be written in the form

$$\sigma = \sigma_{dc} + \delta p \qquad (1 > \delta > 0,\ \sigma > p), \tag{7.7}$$

where σ_{dc} is the strength of the rock with uniform biaxial compression; δ is some empirical constant.

With the help of Eqs. (7.6) and (7.7), we obtain an expression for the limiting drilling depth H_{**} in the case of reinforcement of the well shaft:

$$H_{**} = \frac{\sigma_{dc}}{\rho_H g(1-\delta) + g\sqrt{\rho\rho_H - \frac{1}{4}\rho_H^2 - \frac{1}{2}\rho^2}}.$$

For example, for $\rho = 1.7\rho_H$, $\delta = 1/2$, $\rho_H g = 2\ \text{g/cm}^3$, $\sigma_{dc} = 20\ \text{kg/mm}^2$, $H_{**} = 20$ km. For rocks with lower strength, this depth can be much less. For $H > H_{**}$, self-sustaining fracture of the bottom hole occurs.

In a similar way, it is possible to estimate the stresses on the bottom hole of an unreinforced well, but this estimate is of less interest, since in this case, large stresses occur on the well walls far away from the bottom hole.

The results obtained on stability and cavern formation in rocks can be generalized, assuming that in all of the equations presented above, the quantity σ_c is a function of temperature and loading time; for example, according to the analogy between temperature and time, it is natural to take the following dependence:

$$\sigma_c = \sigma_{c0}\left(1 - \frac{RT}{U}\ln\frac{\tau}{\tau_0}\right),$$

where τ is the time from the moment that the bottom hole passes the volume of rock examined; σ_{c_0} and τ_0, some experimentally determined constants, which do not depend on temperature and time; T, absolute temperature; R, gas constant; U, activation energy. In this case, the loss of stability and cavern formation will begin in the higher sections of the unreinforced well in the homogeneous rock; layered rocks can be taken into account using the same equations.

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PERIODIC PROBLEM OF THE INTERACTION OF SYSTEMS OF CIRCULAR OPENINGS AND STRINGERS

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Problems involving interaction of different types of concentrators, viz., openings, cuts, rigid edges (stringers), arising in technology have been the subject of a number of investigations, which are reviewed in [1, 2]. In particular, the interaction of an opening with one and two stringers was examined in [1, 3], and the interaction of a periodic system of cuts and stringers was examined in [4].

In this paper, we examine the mutual effect of a periodic system of circular openings, situated along a straight line, and a periodic system of stringers, orthogonal to this straight line. In this case, it is important to combine the methods in [1, 4, 5], developed for singular concentrators, with the techniques for solving problems on the weakening of a surface by an opening and a periodic system of openings [6, 7].

We shall examine a plate, consisting of a periodic system of circular openings and a periodic system of stringers (Fig. 1). The centers of the openings γ_k ($k = 0, \pm 1, \pm 2, \ldots$) are situated on the straight line $y = 0$ at the points $x_k = 2kb$, and the radii of the openings equal ρ ($\rho < b$). The stringers Γ_k continuously fixed to the plate have the same length $2a$ ($a < b$), perpendicular to the straight line $y = 0$ and intersected at the points $x_k = (2k + 1)b$. The stringers do not resist bending and function only under tension; E, ν, and h are, respectively, the elastic modulus, Poisson's coefficient, and the thickness of the plate; E_0, and S_0 are the elastic modulus and surface area of the transverse section of a stringer.

For the elements in the elastic fields the following notation is used: σ_x, σ_y, τ_{xy}, stress components; u, v, components of the displacements of the plate; N(y), normal force in the section of a stringer; $\varepsilon^0(y)$, relative elongation of its axis.

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